On the Degrees of Freedom in the Interaction between Sets of Elementary Scatterers

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***Abstract*— A series of numerical experiments has been executed to investigate the number of degrees of freedom in the interaction between large scattering bodies. The relation between degrees of freedom and operating frequency establishes the computational complexity for integral equation methods fast solvers based on matrix block compression. In 2D the theoretical asymptotic relation for large frequency soon becomes clear. On the other hand, in 3D even a tendency towards the asymptotic value fails to appear for bodies of hundreds of wavelengths in diameter.**

I. Introduction

Over the past decades, several methods have been proposed to accelerate the solution of electromagnetic scattering and radiation problems using the Method of Moments (MoM). A common parameter by which the relative merits of these methods are measured is the computational complexity (CC). This is understood as the asymptotic scaling of the operation count (and storage requirements) with the number of unknowns N involved in the problem. Generally, one can distinguish between the CC with respect to discretization size while fixing the problem frequency, and the reverse. These two cases are sometimes referred to as the low and high frequency regimes, respectively. The latter is the more common and more interesting measure for electromagnetic problems since, generally, a given desired accuracy corresponds to a fixed discretization.

The above mentioned acceleration methods are based on approximating the interactions between two sets of approximately n elementary scatterers (basis functions, unknowns) at a 'sufficient' mutual distance. The CC derivations all hinge on the fact that, to a given accuracy, these interactions can be computed in less than the obvious n^2 scalar operations.

Some methods achieve acceleration through a purely mathematical technique, for example AIM [1], which uses the FFT to speed up the matrix-vector products that determine the CC of an iterative solver. AIM yields a CC of $N^{3/2}\log N$ (for surface discretization), independently of any physical property of the kernel of the integral equation. The MLFMA [2] does depend on the kernel, but its CC equals $N\log N$ due to a nested computation of the interactions, similar to an FFT.

Other methods do, for their acceleration, depend on the properties of the kernel, more specifically on the information content, or degrees of freedom, present in the interaction between the two sets. Examples are MLMDA [3] and the ACA [4] method applied to electromagnetics [5]. These methods depend, for their rate of acceleration, on the *compressibility* of the matrix blocks representing interactions between 'far' sets. These blocks are generally rank-deficient and can be approximated with a product of two low rank matrices. The CC of these methods depends on how this rank, or the number of significant degrees of freedom (DoF), scales with the frequency.

According to theory, revised in section II, the DoF scale linearly with the frequency in 2D and quadratically in 3D. For fixed discretization size, restricting ourselves to surface discretization, the necessary number of unknowns scales with the same factor. This limits the obtainable reduction in CC through compression. However, several studies have reported experimental values of the CC that are far lower than the asymptotic value and we have come across none that actually approaches this value. Nor have we ever observed it in our own experiments. For instance, [5] reports a numerical experiment in 3D revealing a CC of $N^{4/3}\log N$, for an iterative method, in the range of $N = 1\times10^4 - 2\times10^5$. The paper observes that for $N \rightarrow \infty$, the CC could degrade to N^2. We do not agree, since, as we showed in [6], even if the asymptotic value of DoF $\sim k^2$ is reached, the optimum choice of subdivision in sets still leads to $N^{3/2}$ complexity. The question remains, however, of when this happens.

The above prompted us to perform a numerical study and try to establish the order of magnitude of problems for which the asymptotic behaviour actually materialises. The results of this study are presented in section III.

II. Theory

In 2D, the scattered field from any distribution of elementary scatterers contained within a circle of radius a is entirely resolved, outside that circle, by a number of cylindrical harmonics proportional to k_0a [3]. Evaluating this field over the surface of a second set contained within a circle of similar size at a minimum distance R from the first requires a number of samples proportional to k_0a times the aperture angle covered by the second set as seen from the centre of the first set. Therefore, the DoF between the two sets are proportional to [3]

$$\mathrm{DoF}_{2D} \sim k\, a^2/R. \qquad (1)$$

In 3D, the domains are delimited by spheres rather than circles, and the fields are expressed as spherical harmonics. For sources contained within a sphere of radius ρ, the necessary number of spherical harmonics is proportional to $(k\rho)^2$ [7]. The necessary number of evaluation points inside a second sphere is proportional to the solid angle covered by this one as seen from the first. Therefore, sets of scatterers within two spheres at a minimum mutual distance R represent

$$\mathrm{DoF}_{3D} \sim k^2\, \rho^4/R^2 \qquad (2)$$

degrees of freedom. Equation (2) is similar to the expression derived for communicating antennas in [8], which is specified to be valid for DoF >> 1. In [9] a corresponding expression is obtained without invoking spherical harmonics but using interpolation theory. The expression in [9] is more general in the sense that the two domains need not be spheres; the factors ρ^2 appear as the surface of convex domains enclosing the elementary scatterers.

As Eqs. (1) and (2) show, the DoF are proportional to k in 2D and to k^2 in 3D, just as the number of unknowns in MoM high-frequency problems with surface discretization. This does not mean that compression is useless for large problems, of course. Firstly, there is an important constant gain (independent of the frequency). Secondly, in [6], we showed that in 3D, due to the factor $1/R^2$ in (2), scaling the set-size with $\sqrt{N}$ yields a CC of $N^{3/2}$ for iterative solvers, even for large N.

III. NUMERICAL STUDY

A reliable way to determine the number of DoF of a system by numerical simulation is to compute the SVD and apply a threshold with respect to the largest singular value [10]. However, we are looking for the asymptotic behaviour for large systems, and the SVD is an expensive algorithm. Fortunately, the ACA yields a good approximation with much less computational effort. The result of the ACA can easily be converted into a truncated SVD decomposition [6]. In our experience, the ACA with a given threshold diverges from the true SVD in the last few singular values. Therefore, we execute the ACA with a threshold τ/10 and convert the result into a truncated SVD with a threshold τ, thus removing the tail of unreliable singular values.

We have started with an investigation of (1) for the 2D case. To this aim, we applied ACA to the interaction matrix of two identical canonical PEC geometries, two flat plates and two circular cylinders, discretized by point-matching and with TM-mode illumination. Accordingly the matrix entries reduced to order-zero, second kind Hankel functions $H_0^{(2)}(kr)$. We used a fixed, large number of points ($N = 10{,}000$ per plate and $N = 20{,}000$ per sphere), and executed the calculations for a range of frequencies. In view of the criterion from [8], that (1) is valid only for DoF >> 1, we chose to position the two sets quite close together, at a distance $D = 3d$ for the plates, where d is the plate width. The circles are at a minimum distance $D_{min} = 3a$, where a is the radius.

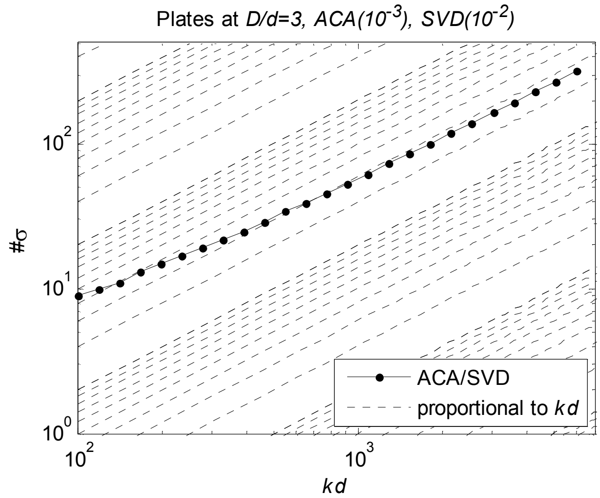

Fig. 1. Number of singular values above SVD threshold for 2D flat plates at broadside, width d, distance $3d$.

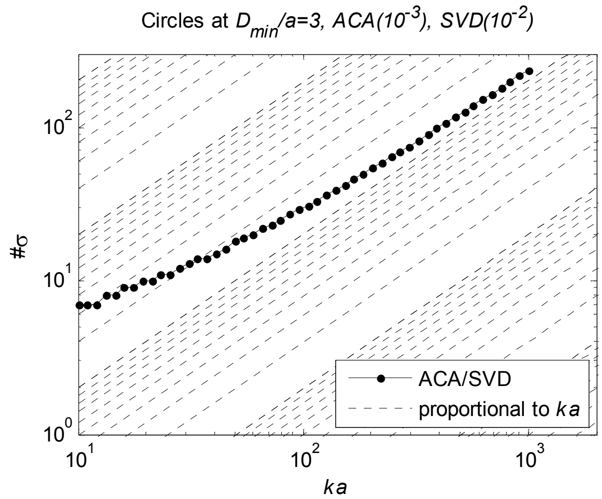

Fig. 2. Number of singular values above SVD threshold for 2D circular cylinders of radius a, distance $D_{min} = 3a$.

Figures 1 and 2 show the number of singular values above a chosen threshold, in casu $\tau = 10^{-2}$. In both cases, the k proportionality predicted by (1) sets in beyond a given electrical size. Around an electrical size of about 150λ, it is practically there. Fig. 3 shows the normalized singular values down to the threshold for the successive flat plate calculations of Fig. 1. It clearly shows a number of significant singular values proportional to kd and then a sudden steep drop, in accordance with the theory.

Subsequently we turned to the 3D case. We analysed two square plates, at broadside, edge length d and mutual distance

3d. First we discretized the plates using $N = 119{,}600$ RWG basis functions per plate. Upon reaching the limit of the computational capacity at our disposal, we repeated the same calculation with the free space scalar Green's function evaluated on a uniform grid, for $N = 160{,}000$ points per plate. This allowed us to go further and at the same time to check whether the results were independent of the discretization. As Fig. 4 shows, this turned out to be the case; the two discretizations correspond very well, apart from a factor two. The factor two reflects the fact that the vectorial RWG formulation incorporates two independent polarizations.

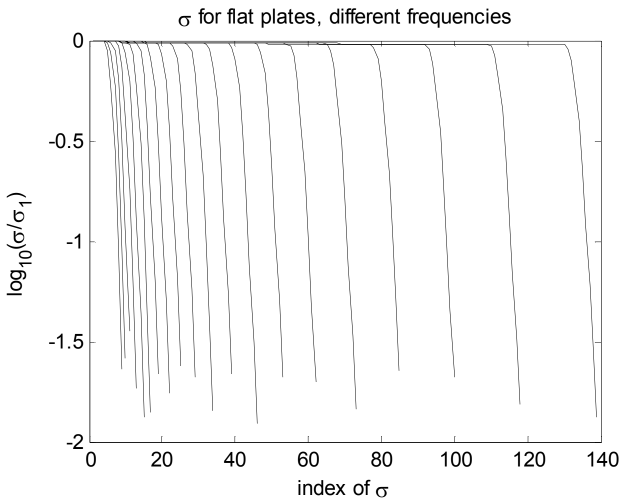

Fig. 3. Distribution of singular values for two flat plates in 2D, for the frequencies of Fig. 1.

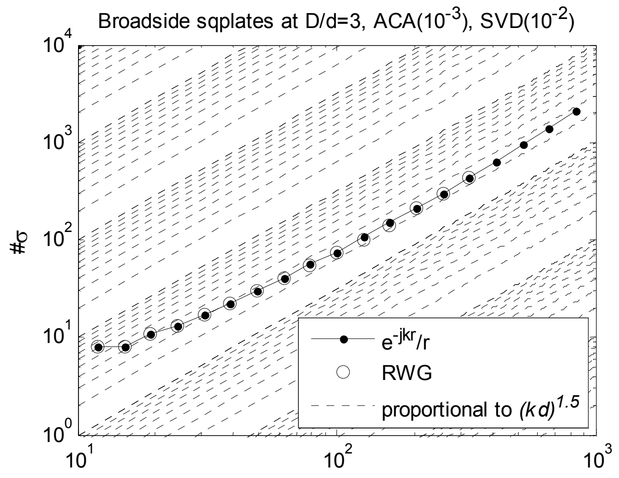

Fig. 4. Number of singular values above SVD threshold for 3D square plates at broadside, width d. The RWG results are divided by two.

Using many more unknowns in 3D we managed to obtain a frequency range that allows comparison with the 2D case. Clearly, the proportionality with the frequency is nowhere near $(kd)^2$ as predicted by (2). Rather it seems to converge around $(kd)^{3/2}$. The distribution of the singular values, shown in Fig. 5, behaves quite differently too: there is a sharp drop immediately after the dominant σ and this drop grows with the frequency. We have at this stage unfortunately no explanation for these observations.

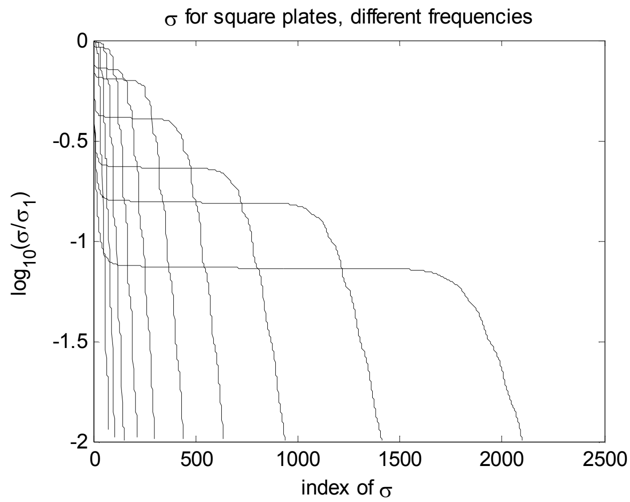

Fig. 5. Distribution of singular values for two square plates in 3D, for the frequencies of Fig. 4.

CONCLUSION

A number of numerical experiments have been presented in this paper, both in 2D and in 3D, that were conducted to establish the asymptotical proportionality of the DoF versus frequency between sets of mutually distant elementary scatterers, as predicted by theory. In 2D the theoretical prediction that the DoF are proportional to the frequency, is confirmed, for moderately (electrically) large scatterers. In 3D, however, the predicted proportionality with the square of the frequency, is not observed for comparable sizes. It appears that in 3D the asymptotical value is only approached when the sets of scatterers are very electrically large. This means that MoM accelerators based on this mechanism are much more efficient than prescribed by the computational complexity.

ACKNOWLEDGMENT

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